Partial Differential Equations

MATH-SHU 263

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Contents

1 Questions

- 1.4 Initial and Boundary Conditions / The Vibrating String [Page 21]

Q1: Why there is no tension at the end implies $u_x = 0$? What is the physical meaning of u_x? What does the Robin Condition mean for the vibrating string?

A1: u_x can be understood as the tension of a unit mass point. Robin Condition for the vibrating string can be understood as fixing a spring at the end of the rope.

- Q2: Can Green's function be used to solve Wave or Diffusion Equations?
- A2:
- Q3: Will we encounter Green's function at our exam?

2 Review[1]

2.1 A General View of Equations With Physical Meanings

1. $u_x + u_y = 0$ (transport)
2. $u_x + yu_y = 0$ (transport)
3. $u_x + uu_y = 0$ (shock wave)
4. $u_{xx} + u_{yy} = 0$ (Laplace's equation)
5. $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)
6. $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)
7. $u_{tt} + u_{xxxx} = 0$ (vibrating bar)
8. $u_t - iu_{xx} = 0$ $(i = \sqrt{-1})$ (quantum mechanics)

2.2 Well-posedness Problem

- **A.** Existence: There exists at least one solution.
- **B.** Uniqueness: There exists at most one solution. For those well-posed problems, maximum principle is usually a good way to prove the uniqueness of the solution. If too few auxiliary conditions are imposed, then there may be more than one solution (nonuniqueness) and the problem is called underdetermined.
- **C.** Stability: The solution is stable within perturbation of the I.C. or B.C. If there are too many auxiliary conditions, there may be no solution at all (nonexistence) and the problem is called

overdetermined.

- Ill-posed Examples:

1) The solution for backward-in-time diffusion equation will blow up at the origin.

2) Consider Laplace's Equation $u_{xx} + u_{yy} = 0$ in the upper half plane $D = \{-\infty < x < \infty, 0 < y < \infty\}$. It is not a well-posed problem to specify both u and u_y on the boundary of D, for the following reason. It has the solutions

$$u_n(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny.$$

Notice that they have boundary data $u_n(x, 0) = 0$ and $\frac{\partial}{\partial y} u_n(x, 0) = e^{-\sqrt{n}} \sin nx$, which tends to zero as $n \to \infty$. But for $y \neq 0$ the solutions $u_n(x, y)$ do not tend to zero as $n \to \infty$ because $\sinh ny$ grows faster than $\frac{1}{n} e^{-\sqrt{n}}$. Thus the stability is violated.

2.3 First-order Linear PDE

- 1. $au_x + bu_y = 0 \Rightarrow u(x, y) = f(bx - ay)$
- 2. $g(x, y)u_x + h(x, y)u_y = 0 \Rightarrow u(x, y) = f(C)$, where C is the constant derived from $\frac{dy}{dx} = \frac{h(x,y)}{g(x,y)}$
- 3. $au_x + bu_y + k(x, y)u = m(x, y)$

The Coordinate Method: Change of Variables

Step 1: let $\alpha = ax + by, \beta = bx - ay$, substitute every x,y by α, β

Step 2: Solve $u(\alpha, \beta)$ (integration factor)

Step 3: Substitute α, β by x,y to get u(x,y)

- 4. General Form: $g(x, y)u_x + h(x, y)u_y + k(x, y)u = m(x, y)$

The Geometric Method: Characteristic Curves

Step 1: $\frac{dy}{dx} = \frac{h(x,y)}{g(x,y)}$, derive the relation between x,y, and the constant E

Step 2: $\frac{du}{dx} + \frac{k(x,y)}{g(x,y)} u = \frac{m(x,y)}{g(x,y)}$, substitute y by x and E, then use the integration factor to solve u(x,E)

Step 3: Substitute E by x and y to find u(x,y)

Example: $u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$

Solution by the Geometric Method: Characteristic Curves:

Step 1: $\frac{dy}{dx} = 2 \Rightarrow y = 2x + E$

Step 2: $\frac{\partial u}{\partial x} + (2x - y)u = 2x^2 + 3xy - 2y^2$

Substituting y by x and E: $\Rightarrow \frac{\partial u}{\partial x} - Eu = -5Ex - 2E^2$

By integration factor $e^{\int -Edx} = e^{-Ex}$, we have $u(x,E) = \frac{\int e^{-Ex}(-5Ex-2E^2)dx+f(E)}{e^{-Ex}} = 2E + 5x + \frac{5}{E} + f(E)e^{Ex}$

Step 3: Substituting E by x and y: $u(x,y) = 2y + x + \frac{5}{y-2x} + f(y-2x)e^{(y-2x)x}$

Solution by the Coordinate Method: Change of Variables:

Step 1: Let $\alpha = x + 2y, \beta = 2x - y$, then $x = \frac{\alpha+2\beta}{5}, y = \frac{2\alpha-\beta}{5}$, $u_x = u_\alpha + 2u_\beta, u_y = 2u_\alpha - u_\beta$
Substituting into the equation: $u_\alpha + \frac{\beta}{5}u = \frac{\alpha\beta}{5}$

Step 2: By integration factor $e^{\int \frac{\beta}{5}d\alpha} = e^{\frac{1}{5}\alpha\beta}$, we have $u(\alpha,\beta) = \frac{\int \frac{1}{5}\alpha\beta e^{\frac{1}{5}\alpha\beta}d\alpha+g(\beta)}{e^{\frac{1}{5}\alpha\beta}} = \alpha - \frac{5}{\beta} + g(\beta)e^{-\frac{1}{5}\alpha\beta}$

Step 3: By substituting α, β by x,y, we get $u(x,y) = x + 2y - \frac{5}{2x-y} + g(2x-y)e^{-\frac{1}{5}(x+2y)(2x-y)}$
Remark: The solutions given by the two methods should coincide, which can be seen by setting $z = 2x + y$ and take $f(-z) = g(z)e^{\frac{2}{5}z^2}$.

2.4 Type of PDE: Elliptic, Hyperbolic, Parabolic

General form of the second-order equations: $a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$

- Elliptic: $a_{12}^2 - a_{11}a_{22} < 0$
- Hyperbolic: $a_{12}^2 - a_{11}a_{22} > 0$
- Parabolic: $a_{12}^2 - a_{11}a_{22} = 0$

2.5 The Wave Equation

- 1. Without I.C. and B.C. on $\mathbb{R}$:

$$u_{tt} = c^2u_{xx}, \text{ for } -\infty < x < +\infty \tag{1}$$

Solution:

$$u(x,t) = f(x+ct) + g(x-ct) \tag{2}$$

Method: Operator Factorization [Page 33-34]

- 2. With I.C. Without B.C. on $\mathbb{R}$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, \text{ for } -\infty < x < +\infty \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) \end{cases} \tag{3}$$

Solution:

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \tag{4}$$

Method: Direct deduction from the general solution [Page 35-36]

Examples: The Pluck String [Page 36-37]

Domain of Influence & Interval of Dependence: [Compare with the solution (4)]

Principle of Causality: No part of wave goes faster than the light speed c.

Let $\phi(x)$ denote the initial position, $\psi(x)$ denote as the initial velocity. An initial condition (position $\phi(x)$, velocity $\psi(x)$ or both) at the point $(x_0, 0)$ can affect the solution for $t > 0$ only in the shaded sector, which is called the domain of influence of the point $(x_0, 0)$. As a consequence, if $\phi(x)$ and $\psi(x)$ vanish for $|x| > R$, then $u(x,t) = 0$ for $|x| > R + ct$. In words, the domain of influence of an interval $|x| \leq R$ is a sector ($|x| \leq R + ct$).

Conversely, fix a point (x_0, t_0) for $t_0 > 0$. $u(x_0, t_0)$ depends only on the values of $\phi(x)$ at the two points $x_0 \pm ct_0$, and depends only on the values of $\psi(x)$ within the interval $[x_0 - ct_0, x_0 + ct_0]$. We therefore say that the interval $(x_0 - ct_0, x_0 + ct_0)$ is the interval of dependence of the point (x_0, t_0) on $t = 0$. Sometimes we call the entire shaded triangle Δ the domain of dependence or the past history of the point (x_0, t_0). The domain of dependence is bounded by the pair of characteristic lines that pass through (x_0, t_0). [Page 39]

The Conservation of Energy: $E = \frac{1}{2}\int_{-\infty}^{+\infty} (\rho u_t^2 + T u_x^2)\, dx$, which is a constant independent of t [Page 40]. Compare it with the Energy in the Laplace Equation (35).

- 3. With I.C. and Dirichlet B.C. on $\mathbb{R}^+$

$$\begin{cases} u_{tt} = c^2 u_{xx}, \text{ for } 0 < x < +\infty \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), \text{ for } 0 < x < +\infty \\ u(0,t) = 0 \end{cases} \tag{5}$$

Solution:

$$u(x,t)=\frac{1}{2}\left[\phi_{\text{odd}}\,(x+ct)+\phi_{\text{odd}}\,(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{\text{odd}}\,(y)dy,\ \text{ for } x\geq 0$$
$$=\begin{cases}\frac{1}{2}[\phi(x+ct)+\phi(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy, \text{ for } x>c|t|.\\ \frac{1}{2}[\phi(ct+x)-\phi(ct-x)]+\frac{1}{2c}\int_{ct-x}^{ct+x}\psi(y)dy, \text{ for } 0<x\leq c|t|.\end{cases}$$

Method: Odd Extension for $\phi(x)$ and $\psi(x)$ [Page 61-63]

- 4. With I.C. and Dirichlet B.C. on $[0,\,\ell]$

$$\begin{cases}u_{tt}=c^2u_{xx},\ \text{ for } 0<x<\ell\\ u(x,0)=\phi(x),\, u_t(x,0)=\psi(x),\ \text{ for } 0<x<\ell\\ u(0,t)=u(\ell,t)=0\end{cases}\tag{6}$$

Solution (d'Alembert):

$$u(x,t)=\frac{1}{2}\phi_{\text{ext}}(x+ct)+\frac{1}{2}\phi_{\text{ext}}(x-ct)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{\text{ext}}(s)ds, \text{ for } 0\leq x\leq\ell$$

Method: Periodic Odd Extension for $\phi(x)$ and $\psi(x)$ [Page 63-66, complicated]

$$(\psi)\,\phi_{\text{ext}}\,(x)=\begin{cases}\phi(x) & \text{for} \quad 0<x\leq\ell\\ -\phi(-x) & \text{for} \quad -\ell<x\leq 0\\ \text{ extended to be of period } 2\ell. & \end{cases}\tag{7}$$

- 5. With I.C. Without B.C. Inhomogeneous Wave on $\mathbb{R}$ (with a source) [See Section 3.4, Page 71-78]

$$\begin{cases}u_{tt}-c^2u_{xx}=f(x,t) \text{ for } -\infty<x<+\infty\\ u(x,0)=\phi(x),\, u_t(x,0)=\psi(x)\end{cases}\tag{8}$$

Solution:
$$u(x,t)=\frac{1}{2}[\phi(x+ct)+\phi(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi+\frac{1}{2c}\iint_{\Delta}f$$

where Δ is the characteristic triangle.

2.6 The Diffusion Equation

- 1. With I.C. Without B.C. on $\mathbb{R}$ [Page 46-52]

$$\begin{cases} u_t = ku_{xx} \quad (-\infty < x < +\infty, 0 < t < +\infty) \\ u(x,0) = \phi(x) \end{cases} \tag{9}$$

Solution:

$$u(x,t) = S * \phi(x) = \int_{-\infty}^{+\infty} S(x-y,t)\phi(y)dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)dy \tag{10}$$

$$= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4}\phi(x - p\sqrt{kt})dp \text{ , by letting } p = (x-y)/\sqrt{kt} \tag{11}$$

where $S(x,t) = \frac{\partial Q(x,t)}{\partial x} = \frac{1}{2\sqrt{\pi kt}} e^{-x^2/4kt}$ for $t > 0$, which is usually called source function, Green's function, gaussian, or diffusion kernel.

Remark: The special initial condition for Q(x,t): $Q(x,0) = 1$ for $x > 0$ and $Q(x,0) = 0$ for $x < 0$, is to set its initial stage as a Heaviside function. [See Page 335]

Some properties:

- 1. $\int_{-\infty}^{\infty} S(x,t)dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2} dq = 1$, by letting q $= \frac{x}{2\sqrt{kt}}$.
- 2. $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy = \int_{-\infty}^{\infty} S(z,t)\phi(x-z)dz$
- 3. $S(x,0) = \delta(x)$, $\delta * f(x) = f(x)$

Theorem 2.6.1. Let $\phi(x)$ be a bounded piecewise continuous function for $-\infty < x < \infty$. Then the formula (11) defines an infinitely differentiable function $u(x,t)$ for $-\infty < x < \infty, 0 < t < \infty$, which satisfies the equation $u_t = ku_{xx}$ and $\lim_{t \searrow 0} u(x,t) = \phi(x)$ for each x. [Page 81-83]

- 2. With I.C. and Dirichlet B.C. on $\mathbb{R}^+$

$$\begin{cases} u_t = ku_{xx} \quad (-\infty < x < +\infty, 0 < t < +\infty) \\ u(x,0) = \phi(x) \\ u(0,t) = 0 \end{cases} \tag{12}$$

Solution:

$$\begin{aligned} u(x,y) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)dy, \text{ for } x>0 \\ &= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \right] \phi(y)dy. \end{aligned}$$

Method: Odd Reflection [Page 57-60]

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x>0 \\ -\phi(-x) & \text{for } x<0 \\ 0 & \text{for } x=0 \end{cases} \tag{13}$$

- 3. With I.C. and Neumann B.C. on $\mathbb{R}^+$

$$\begin{cases} u_t = ku_{xx} \quad (-\infty < x < +\infty, 0 < t < +\infty) \\ u(x,0) = \phi(x) \\ u_x(0,t) = 0 \end{cases} \tag{14}$$

Solution:

$$\begin{aligned} u(x,y) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)dy, \text{ for } x>0 \\ &= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right] \phi(y)dy. \end{aligned}$$

Method: Even Reflection [Page 59-60]

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{for } x \geq 0 \\ \phi(-x) & \text{for } x \leq 0 \end{cases} \tag{15}$$

- 4. With I.C. Without B.C. Inhomogeneous Diffusion on $\mathbb{R}$ (with a source) [Page 67-69]

$$\begin{cases} u_t - ku_{xx} = f(x,t) \quad (-\infty < x < +\infty, 0 < t < +\infty) \\ u(x,0) = \phi(x) \end{cases} \tag{16}$$

Solution: $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds$.

- 5. With I.C. and Dirichlet B.C. Inhomogeneous Diffusion on $\mathbb{R}^+$ [Page 70, Ex. 3.3.2]

$$\begin{cases} u_t - ku_{xx} = f(x,t) \quad (0 < x < +\infty, 0 < t < +\infty) \\ u(x,0) = \phi(x) \\ u(0,t) = h(t) \end{cases} \tag{17}$$

Solution:

$$\begin{aligned} u(x,t) &= h(t) + \int_{-\infty}^{+\infty} S(x-y,t)\Phi_{\text{odd}}(y)dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)F_{\text{odd}}(y,s)dyds, \text{ for x} \geq 0 \\ &= h(t) + \int_0^{+\infty} [S(x-y,t) - S(x+y,t)]\phi(y)dy \\ &+ \int_0^t \int_0^{+\infty} [S(x-y,t-s) - S(x+y,t-s)]f(y,s)dyds \\ &+ \int_0^t \int_0^{+\infty} \frac{\partial}{\partial s}[S(x-y,t-s) - S(x+y,t-s)]h(s)dyds, \text{ for x} \geq 0 \end{aligned}$$

$$\text{where } \Phi_{\text{odd}}(x) = \begin{cases} \phi(x) - h(0), & x \geqslant 0 \\ -\phi(-x) + h(0), & x < 0 \end{cases}, \quad F_{\text{odd}}(x,t) = \begin{cases} f(x,t) - h'(t), & x \geqslant 0 \\ -f(-x,t) + h'(t), & x < 0 \end{cases}$$

Method: Take $v(x,t) = u(x,t) - h(t)$ and use Odd Reflection

- 6. With I.C. and Neumann B.C. Inhomogeneous Diffusion on $\mathbb{R}^+$ [Page 70, Ex. 3.3.3]

$$\begin{cases} u_t - ku_{xx} = f(x,t) \quad (0 < x < +\infty, 0 < t < +\infty) \\ u(x,0) = \phi(x) \\ u_x(0,t) = h(t) \end{cases} \tag{18}$$

Solution:

$$\begin{aligned} u(x,t) &= xh(t) + \int_{-\infty}^{+\infty} S(x-y,t)\Phi_{\text{even}}(y)dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)F_{\text{even}}(y,s)dyds, \text{ for x} \geq 0 \\ &= xh(t) + \int_0^{+\infty} [S(x-y,t) + S(x+y,t)]\phi(y)dy \\ &+ \int_0^t \int_0^{+\infty} \frac{\partial}{\partial s}[S(x-y,t-s) + S(x+y,t-s)]yh(s)dyds, \text{ for x} \geq 0 \end{aligned}$$

where $\Phi_{\text{even}}(x)=\begin{cases}\phi(x)-xh(0), & x\geqslant 0\\ \phi(-x)-xh(0), & x<0\end{cases}, \quad F_{\text{even}}(x,t)=\begin{cases}-xh'(t), & x\geqslant 0\\ xh'(t), & x<0\end{cases}$

Method: Take $v(x,t)=u(x,t)-xh(t)$ and use Even Reflection

- 7. Diffusion with Constant Dissipation [Ex. 2.4.16]
- 8. Diffusion with Variable Dissipation [Ex. 2.4.17]
- 9. Heat with Convection [Ex. 2.4.18]

2.7 The Laplace & Poisson's Equation

Remark: The Laplace Equation is the stationary state of the Wave Equation or Diffusion Equation. Thus it has no I.C.'s. The inhomogeneous Laplace Equation is named Poisson's Equation. In this section, we introduce the solution of the Laplace Equation with Green's Function. We put the Fourier Series solutions & Poisson's Formula to the Fourier Series section.

- 1. Poisson's Equation with Inhomogeneous Dirichlet B.C.

$$\begin{cases}\Delta u=f \text{ in D}\\ u=h \text{ on bdy D.}\end{cases} \tag{19}$$

Solution:

$$u(\mathbf{x}_0)=\iint_{\text{bdy } D} h(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial n}dS+\iiint_D f(\mathbf{x})G(\mathbf{x},\mathbf{x}_0)\,d\mathbf{x}. \tag{20}$$

where $G(x,x_0)$ is the Green's function (2.7.2) for Dirichlet Poisson's Equation.

- 2. Representation Formula

$$u(\mathbf{x}_0)=\iint_{\text{bdy } D}\left[-u(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right)+\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\frac{\partial u}{\partial n}\right]\frac{dS}{4\pi} \tag{21}$$

Proof. The representation formula (21) is the special case of Green's Second Identity (30) with the choice $v(\mathbf{x})=(-4\pi|\mathbf{x}-\mathbf{x}_0|)^{-1}$. Clearly, the right side of (30) agrees with (21). Also, $\Delta u=0$ and $\Delta v=0$, which kills the left side of (30). So where does the left side of (21) come from? From the fact that the function $v(\mathbf{x})$ is infinite at the point $\mathbf{x}_0$. Therefore, it is forbidden to apply (30) in the whole domain D. So let's take a pair of scissors and cut out a small ball around $\mathbf{x}_0$. Let D_ϵ be the region D with this ball (of radius ϵ and center $\mathbf{x}_0$) excised.

For simplicity let $\mathbf{x}_0$ be the origin. Then $v(\mathbf{x}) = -1/(4\pi r)$, where $r = (x^2 + y^2 + z^2)^{1/2} = |\mathbf{x}|$. Writing down (30) with this choice of v, we have, since $\Delta u = 0 = \Delta v$ in D_ϵ,

$$-\iint_{\text{bdy}} \left[u \cdot \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial n} \cdot \frac{1}{r} \right] dS = 0.$$

But bdy D_ϵ consists of two parts: the original boundary bdy D and the sphere $\{r = \epsilon\}$. On the sphere, $\partial/\partial n = -\partial/\partial r$. Thus the surface integral breaks into two pieces,

$$-\iint_{\text{bdy } D} \left[u \cdot \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial n} \cdot \frac{1}{r} \right] dS = -\iint_{r=\epsilon} \left[u \cdot \frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial r} \cdot \frac{1}{r} \right] dS. \tag{22}$$

This identity (22) is valid for any small $\epsilon > 0$. Our representation formula (21) would follow provided that we could show that the right side of (22) tended to $4\pi u(\mathbf{0})$ as $\epsilon \to 0$ Now, on the little spherical surface $\{r = \epsilon\}$, we have

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2} = -\frac{1}{\epsilon^2}$$

so that the right side of (22) equals

$$\frac{1}{\epsilon^2} \iint_{r=\epsilon} u dS + \frac{1}{\epsilon} \iint_{r=\epsilon} \frac{\partial u}{\partial r} dS = 4\pi \bar{u} + 4\pi\epsilon \overline{\frac{\partial u}{\partial r}}, \tag{23}$$

where $\bar{u}$ denotes the average value of $u(\mathbf{x})$ on the sphere $|\mathbf{x}| = r = \epsilon$, and $\overline{\partial u/\partial r}$ denotes the average value of $\partial u/\partial n$ on this sphere. As $\epsilon \to 0$, the expression (34) approaches

$$4\pi u(\mathbf{0}) + 4\pi \cdot 0 \cdot \frac{\partial u}{\partial r}(\mathbf{0}) = 4\pi u(\mathbf{0})$$

because u is continuous and $\partial u/\partial r$ is bounded. Thus (22) turns into (21), and this completes the proof.

□

2.7.1 Properties of the Laplace Operator

Theorem 2.7.1. [Infinite Differentiability of Harmonic Functions][Page 170]

Let u be a harmonic function in any open set D of the plane. Then $u(\mathbf{x}) = u(x, y)$ possesses all partial derivatives of all orders in D.

Proof. This means that $\partial u/\partial x, \partial u/\partial y, \partial^2 u/\partial x^2, \partial^2 u/\partial x\partial y, \partial^{100} u/\partial x^{100}$, and so on, exist automatically. Let's show this first for the case where D is a disk with its center at the origin. Look at Poisson's formula in its second form (56). The integrand is differentiable to all orders for $\mathbf{x} \in D$. Note that $\mathbf{x}' \in \text{bdy } D$ so that $\mathbf{x} \neq \mathbf{x}'$. By the theorem about differentiating integrals (Section A.3), we can differentiate under the integral sign. So $u(\mathbf{x})$ is differentiable to any order in D.
Second, let D be any domain at all, and let $\mathbf{x}_0 \in D$. Let B be a disk contained in D with center at $\mathbf{x}_0$. We just showed that $u(\mathbf{x})$ is differentiable inside B, and hence at $\mathbf{x}_0$. But $\mathbf{x}_0$ is an arbitrary point in D. So u is differentiable (to all orders) at all points of D. $\square$

Remark: This differentiability property is similar to the one we saw in Section 3.5 for the one-dimensional diffusion equation, but of course it is not at all true for the wave equation.

- **[Invariance in 2-D and 3-D]**

The Laplace equation is invariant under all rigid motions.

Proof. **[2-D]**

A rigid motion in the plane consists of translations and rotations. A translation in the plane is a transformation

$$x' = x + a \quad y' = y + b.$$

Invariance under translations means simply that $u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}$. A rotation in the plane through the angle α is given by

$$x' = x\cos\alpha + y\sin\alpha$$

$$y' = -x\sin\alpha + y\cos\alpha$$

By the chain rule we calculate

$$u_x = u_{x'}\cos\alpha - u_{y'}\sin\alpha$$

$$u_y = u_{x'}\sin\alpha + u_{y'}\cos\alpha$$

$$u_{xx} = (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{x'}\cos\alpha - (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{y'}\sin\alpha$$

$$u_{yy} = (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{x'}\sin\alpha + (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{y'}\cos\alpha.$$

Adding, we have

$$u_{xx} + u_{yy} = (u_{x'x'} + u_{y'y'})\left(\cos^2\alpha + \sin^2\alpha\right) + u_{x'y'}.$$
$$= u_{x'x'} + u_{y'y'}.$$

□

This proves the invariance of the Laplace operator. In engineering the Laplacian Δ is a model for isotropic physical situations, in which there is no preferred direction. Also, it suggests that laplacian has polar coordinate form, which is:

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad \text{[Page 156-157]} \tag{24}$$

Remark: For those harmonic functions that themselves are rotationally invariant, by (24), $u = c_1 \log r + c_2$, where $\log r$ plays a central role since it contains a singularity at the origin, otherwise by maximum principle (2.8.2), u would be a constant.

Proof. **[3-D]**

Any rotation in three dimensions is given by

$$\mathbf{x}' = B\mathbf{x}$$

where B is an orthogonal matrix ($BB^\intercal = B^\intercal B = I$). The laplacian is $\Delta u = \Sigma_{i=1}^{3} u_{ii} = \Sigma_{i,j=1}^{3} \delta_{ij} u_{ij}$ where the subscripts on u denote partial derivatives, and $\delta_{ij} = 1$ for i=j; $\delta_{ij} = 0$ for $i \neq j$. Hence

$$\Delta u = \sum_{k,l}\left(\sum_{i,j} b_{ki}\delta_{ij}b_{lj}\right) u_{k'l'} = \sum_{k,l}\delta_{kl}u_{k'l'}$$
$$= \sum_{k} u_{k'k'}$$

because the new coefficient matrix is

$$\sum_{i,j} b_{ki}\delta_{ij}b_{lj} = \sum_{i} b_{ki}b_{li} = (B^\intercal B)_{kl} = \delta_{kl}.$$

So in the primed coordinates Δu takes the usual form

$$\Delta u = u_{x'x'} + u_{y'y'} + u_{z'z'}.$$

□

Also, the polar form in 3-D would be:

$$\Delta_3 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right] \tag{25}$$

Alternatively,

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \quad \text{[Page 158-159]} \tag{26}$$

Remark: For those harmonic functions that themselves are rotationally invariant, by (26), $u = -c_1\frac{1}{\mathbf{r}} + c_2$, where $\frac{1}{\mathbf{r}} = (x^2+y^2+z^2)^{-1/2}$ which is the analog to the 2-D situation. Its laplacian is delta function. In electrostatics the function $u(\mathbf{x}) = \frac{1}{\mathbf{r}}$ is the electrostatic potential when a unit charge is placed at the origin.

2.7.2 Green's First & Second Identity and Green's Function

- Notations:

1. $\operatorname{grad} f \triangleq \nabla f =$ the vector (f_x, f_y, f_z)
2. $\operatorname{div}\mathbf{F} \triangleq \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
3. $\Delta u \triangleq \operatorname{div}\operatorname{grad} u \triangleq \nabla\cdot\nabla u = u_{xx} + u_{yy} + u_{zz}$
4. $|\nabla u|^2 \triangleq |\operatorname{grad} u|^2 = u_x^2 + u_y^2 + u_z^2$

- Gauss's Divergence Theorem

$$\iiint_D \nabla\cdot\mathbf{F}d\mathbf{x} \triangleq \iiint_D \operatorname{div}\mathbf{F}d\mathbf{x} = \iint_{\text{bdy } D} \mathbf{F}\cdot\mathbf{n}\,dS \tag{27}$$

- If we take $\mathbf{F} = \nabla u$, then

$$\iint_{\text{bdy } D} \frac{\partial u}{\partial n}dS = \iiint_D \Delta u d\mathbf{x}. \tag{28}$$

- Green's First Identity

$$\iint_{\text{bdy } D} v(\nabla u\cdot\mathbf{n})dS \triangleq \iint_{\text{bdy } D} v\frac{\partial u}{\partial n}dS = \iiint_D \nabla v\cdot\nabla u d\mathbf{x} + \iiint_D v\Delta u d\mathbf{x} \tag{29}$$

Proof. The Green's First Identity is proved by integrating the property $\nabla\cdot(v\nabla u) = \nabla v\cdot\nabla u + v\Delta u$

and use Gauss's Divergence Theorem. □

- Green's Second Identity

$$\iiint_D (u\Delta v - v\Delta u)d\mathbf{x} = \iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \triangleq \iint_{\text{bdy } D} \left[u(\nabla u \cdot \mathbf{n}) - v(\nabla u \cdot \mathbf{n})\right] dS \tag{30}$$

Proof. The Green's Second Identity is proved by switching u,v in the Green's First Identity and subtract. □

Definition 2.7.2. The **Green's function** $G(\mathbf{x})$ for the operator $-\Delta$ and the domain D at the point $\mathbf{x}_0 \in D$ is a function defined for $\mathbf{x} \in D$ such that:
(i) $G(\mathbf{x})$ possesses continuous second derivatives and $\Delta G = 0$ in D, except at the point $\mathbf{x} = \mathbf{x}_0$.
(ii) $G(\mathbf{x}) = 0$ for $x \in$ bdy D.
(iii) The function $G(\mathbf{x}) + 1/\left(4\pi\left|\mathbf{x} - \mathbf{x}_0\right|\right)$ is finite at $\mathbf{x}_0$ and has continuous second derivatives everywhere and is harmonic at $\mathbf{x}_0$.

Remark: It can be shown that a Green's function exists. Also, it is unique by [Ex.7.3.1]. The usual notation for the Green's function is $G\left(\mathbf{x}, \mathbf{x}_0\right)$.

- See how to find the Green's function for different domain D at Sect. 7.4 [Page 191-196].

Proposition 2.7.3. Symmetricity of Green's function: $G\left(\mathbf{x}, \mathbf{x}_0\right) = G\left(\mathbf{x}_0, \mathbf{x}\right),$ for $\mathbf{x} \neq \mathbf{x}_0$. [Page 189-190]

Proof. We apply Green's second identity (30) to the pair of functions $u(\mathbf{x}) = G(\mathbf{x}, \mathbf{a})$ and $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{b})$ and to the domain D_ϵ. By D_ϵ we denote the domain D with two little spheres of radii ϵ cut out around the points $\mathbf{a}$ and $\mathbf{b}$. So the boundary of D_ϵ consists of three parts: the original boundary bdy D and the two spheres $|\mathbf{x} - \mathbf{a}| = \epsilon$ and $|\mathbf{x} - \mathbf{b}| = \epsilon$. Thus

$$\iiint_{D_\epsilon} (u\Delta v - v\Delta u)d\mathbf{x} = \iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS + A_\epsilon + B_\epsilon, \tag{31}$$

where

$$A_\epsilon = \iint_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$

and B_ϵ is given by the same formula at $\mathbf{b}$. Because both u and v are harmonic in D_ϵ, the left side of (31) vanishes. Since both u and v vanish on bdy D, the integral over bdy D also vanishes.

Therefore, $A_\epsilon + B_\epsilon = 0$, for each ϵ. Let's calculate the limits as $\epsilon \to 0$. We shall then have $\lim A_\epsilon + \lim B_\epsilon = 0$. For A_ϵ, denote $r = |\mathbf{x} - \mathbf{a}|$. Then

$$\lim_{\epsilon \to 0} A_\epsilon = \lim_{\epsilon \to 0} \iint_{r=\epsilon} \left\{ \left(-\frac{1}{4\pi r} + H \right) \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left(-\frac{1}{4\pi r} + H \right) \right\} r^2 \sin\theta d\theta d\phi$$

where θ and ϕ are the spherical angles for $\mathbf{x} - \mathbf{a}$, and H is a continuous function. Now $\partial/\partial n = -\partial/\partial r$ for the sphere. Among the four terms in the last integrand, only the third one contributes a nonzero expression to the limit [for the same reason as in the derivation of (21)]. Thus

$$\lim_{\epsilon \to 0} A_\epsilon = \lim_{\epsilon \to 0} \int_0^{2\pi} \int_0^{\pi} v \frac{1}{4\pi\epsilon^2} \epsilon^2 \sin\theta d\theta d\phi = v(\mathbf{a})$$

by cancellation of the ϵ^2. A quite similar calculation shows that $\lim B_\epsilon = -u(\mathbf{b})$. Therefore,

$$0 = \lim (A_\epsilon + B_\epsilon) = v(\mathbf{a}) - u(\mathbf{b}) = G(\mathbf{a}, \mathbf{b}) - G(\mathbf{b}, \mathbf{a}).$$

This proves the symmetricity. □

2.7.3 Mean Value Property

- [2-dimension]

Theorem 2.7.4. Let u be a harmonic function in a disk D, continuous in its closure $\bar{D}$. Then the value of u at the center of D equals the average of u on its circumference. [Page 169]

Proof. Choose coordinates with the origin $\mathbf{0}$ at the center of the circle. Put $\mathbf{x} = \mathbf{0}$ in Poisson's formula (56), or else put $r = 0$ in (55). Then

$$u(\mathbf{0}) = \frac{a^2}{2\pi a} \int_{|\mathbf{x}'|=a} \frac{u(\mathbf{x}')}{a^2} ds'.$$

This is the average of u on the circumference $|\mathbf{x}'| = a$. □

- [n-dimension]

Theorem 2.7.5. In three dimensions the mean value property states that the average value of any harmonic function over any sphere equals its value at the center. [Page 180-181]

Proof. To prove this statement, let D be a ball, $\{|\mathbf{x}| < a\}$, say; that is, $\{x^2 + y^2 + z^2 < a^2\}$. Then bdy D is the sphere (surface) $\{|\mathbf{x}| = a\}$. Let $\Delta u = 0$ in any region that contains D and bdy D. For a sphere, n points directly away from the origin, so that

$$\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u = \frac{\mathbf{x}}{r} \cdot \nabla u = \frac{x}{r} u_x + \frac{y}{r} u_y + \frac{z}{r} u_z = \frac{\partial u}{\partial r} \quad \left(\frac{\partial x}{\partial r} = \frac{x}{r}, \frac{\partial y}{\partial r} = \frac{y}{r}, \frac{\partial z}{\partial r} = \frac{z}{r}\right)$$

where $r = (x^2 + y^2 + z^2)^{1/2} = |\mathbf{x}|$ is the spherical coordinate, the distance of the point (x, y, z) from the center $\mathbf{0}$ of the sphere. Therefore, (28) becomes

$$\iint_{\text{bdy } D} \frac{\partial u}{\partial r} dS = 0 \tag{32}$$

Let's write this integral in spherical coordinates, (r, θ, ϕ), explicitly. Then (32) takes the form

$$\int_0^{2\pi} \int_0^{\pi} u_r(a, \theta, \phi) a^2 \sin\theta d\theta d\phi = 0$$

since $r = a$ on bdy D. We divide this by the constant $4\pi a^2$ (the area of bdy D). This result is valid for all $a > 0$, so that we can think of a as a variable and call it r. Then we pull $\partial / \partial r$ outside the integral (see Section A.3), obtaining

$$\frac{\partial}{\partial r}\left[\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \phi) \sin\theta d\theta d\phi\right] = 0.$$

Thus

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \phi) \sin\theta d\theta d\phi$$

is independent of r. This expression is precisely the average value of u on the sphere $\{|\mathbf{x}| = r\}$. In particular, if we let $r \to 0$, we get

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(\mathbf{0}) \sin\theta d\theta d\phi = u(\mathbf{0})$$

That is,

$$\frac{1}{\text{area of } S} \iint_S u dS = u(\mathbf{0}) \tag{33}$$

This proves the mean value property in three dimensions. Actually, we can extend it to n dimension.

□

2.7.4 Dirichlet Principle

This is an important mathematical theorem based on the physical idea of energy. It states that among all the functions $w(\mathbf{x})$ in D that satisfy the Dirichlet boundary condition

$$w = h(\mathbf{x}) \quad \text{on bdy } D, \tag{34}$$

the lowest energy occurs for the harmonic function satisfying (34). In the present context the energy is defined as

$$E[w] = \frac{1}{2} \iiint_D |\nabla w|^2 d\mathbf{x}. \tag{35}$$

This is the pure potential energy, there being no kinetic energy because there is no motion. Now it is a general principle in physics that any system prefers to go to the state of lowest energy, called the ground state. Thus the harmonic function is the preferred physical stationary state. Mathematically, Dirichlet's principle can be stated precisely as follows:

Theorem 2.7.6. Let $u(\mathbf{x})$ be the unique harmonic function in D that satisfies (34). Let $w(\mathbf{x})$ be any function in D that satisfies (34). Then

$$E[w] \geq E[u].$$

Proof 1. To prove Dirichlet's principle, we let $v = u - w$ and expand the square in the integral

$$\begin{aligned} E[w] &= \frac{1}{2} \iiint_D |\nabla (u - v)|^2 d\mathbf{x} \\ &= E[u] - \iiint_D \nabla u \cdot \nabla v d\mathbf{x} + E[v]. \end{aligned} \tag{36}$$

Next we apply Green's first identity (29) to the pair of functions u and v. In (29) two of the three terms are zero because $v = 0$ on bdy D and $\Delta u = 0$ in D. Therefore, the middle term in (36) is also zero. Thus

$$E[w] = E[u] + E[v].$$

Since it is obvious that $E[v] \geq 0$, we deduce that $E[w] \geq E[u]$. This means that the energy is smallest when $w = u$. This proves Dirichlet's principle. □

Proof 2. An alternative proof goes as follows. Let $u(\mathbf{x})$ be a function that satisfies (34) and minimizes the energy (35). Let $v(\mathbf{x})$ be any function that vanishes on bdy D. Then $u + \epsilon v$ satisfies the boundary condition (34). So if the energy is smallest for the function u, we have

$$E[u] \leq E[u + \epsilon v] = E[u] - \epsilon \iiint_D \Delta u v d\mathbf{x} + \epsilon^2 E[v]$$

for any constant ϵ. The minimum occurs for $\epsilon = 0$. By calculus,

$$\iiint_D \Delta u v \, d\mathbf{x} = 0 \tag{37}$$

This is valid for practically all functions v in D. Let D' be any strict subdomain of D; that is, $\overline{D'} \subset D$. Let $v(\mathbf{x}) \equiv 1$ for $\mathbf{x} \in D'$ and $v(\mathbf{x}) \equiv 0$ for $\mathbf{x} \in D - D'$. In (37) we choose this function v. (Because this v is not smooth, an approximation argument is required that is omitted here.) Then (37) takes the form

$$\iiint_{D'} \Delta u d\mathbf{x} = 0 \quad \text{for all } D'.$$

By the second vanishing theorem in Section A.1, it follows that $\Delta u = 0$ in D. Thus $u(\mathbf{x})$ is a harmonic function. By uniqueness, it is the only function satisfying (34) that can minimize the energy. □

2.8 Maximum Principle (Diffusion and Laplace)

- 1. Diffusion Equation

Theorem 2.8.1. If $u(x, t)$ satisfies the diffusion equation in a rectangle (say, $0 \leq x \leq l, 0 \leq t \leq T$) in space-time, then the maximum value of $u(x, t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$).

In fact, there is a stronger version of the maximum principle which asserts that the maximum cannot be assumed anywhere inside the rectangle but only on the bottom or the lateral sides (unless u is a constant). The corners are allowed. The minimum value has the same property; it too can be attained only on the bottom or the lateral sides. To prove the minimum principle, just apply the maximum principle to $[-u(x, t)]$.

Proof. We'll prove only the weaker version. (Surprisingly, its strong form is much more difficult to prove.) The idea of the proof is to use the fact, from calculus, that at an interior maximum the first derivatives vanish and the second derivatives satisfy inequalities such as $u_{xx} \leq 0$. If we knew that $u_{xx} \neq 0$ at the maximum (which we do not), then we'd have $u_{xx} < 0$ as well as $u_t = 0$, so that $u_t \neq k u_{xx}$. This contradiction would show that the maximum could only be somewhere on the boundary of the rectangle. However, because u_{xx} could in fact be equal to zero, we need to play a mathematical game to make the argument work.

So let M denote the maximum value of $u(x,t)$ on the three sides $t = 0$, $x = 0$, and $x = l$. (Recall that any continuous function on any bounded closed set is bounded and assumes its maximum on that set.) We must show that $u(x,t) \leq M$ throughout the rectangle R.

Let ϵ be a positive constant and let $v(x,t) = u(x,t) + \epsilon x^2$. Our goal is to show that $v(x,t) \leq M + \epsilon l^2$ throughout R. Once this is accomplished, we'll have $u(x,t) \leq M + \epsilon\left(l^2 - x^2\right)$. This conclusion is true for any $\epsilon > 0$. Therefore, $u(x,t) \leq M$ throughout R, which is what we are trying to prove.

Now from the definition of v, it is clear that $v(x,t) \leq M + \epsilon l^2$ on $t = 0$, on $x = 0$, and on $x = l$. This function v satisfies

$$v_t - k v_{xx} = u_t - k\left(u + \epsilon x^2\right)_{xx} = u_t - k u_{xx} - 2\epsilon k = -2\epsilon k < 0,$$

which is the "diffusion inequality." Now suppose that $v(x,t)$ attains its maximum at an interior point (x_0, t_0). That is, $0 < x_0 < l, 0 < t_0 < T$. By ordinary calculus, we know that $v_t = 0$ and $v_{xx} \leq 0$ at (x_0, t_0). This contradicts the diffusion inequality. So there can't be an interior maximum. Suppose now that $v(x,t)$ has a maximum (in the closed rectangle) at a point on the top edge $\{t_0 = T \text{ and } 0 < x < l\}$. Then $v_x(x_0, t_0) = 0$ and $v_{xx}(x_0, t_0) \leq 0$, as before. Furthermore, because $v(x_0, t_0)$ is bigger than $v(x_0, t_0 - \delta)$, we have

$$v_t(x_0, t_0) = \lim \frac{v(x_0, t_0) - v(x_0, t_0 - \delta)}{\delta} \geq 0$$

as $\delta \to 0$ through positive values. (This is not an equality because the maximum is only "one-sided" in the variable t.) We again reach a contradiction to the diffusion inequality. □

- 2. Laplace Equation

Theorem 2.8.2. Let D be a connected bounded open set (in either two- or three-dimensional space). Let either $u(x,y)$ or $u(x,y,z)$ be a harmonic function in D that is continuous on $\bar{D} = D \cup$ (bdy D). Then the maximum and the minimum values of u are attained on bdy D and nowhere inside (unless $u \equiv$ constant).

Proof. **(n-D Weak)**

In other words, a harmonic function is its biggest somewhere on the boundary and its smallest somewhere else on the boundary.

To understand the maximum principle, let us use the vector shorthand $\mathbf{x}=(x, y)$ in two dimensions or $\mathbf{x}=(x, y, z)$ in three dimensions. Also, the radial coordinate is written as $|\mathbf{x}|=\left(x^2+y^2\right)^{1/2}$ or $|\mathbf{x}|=\left(x^2+y^2+z^2\right)^{1/2}$. The maximum principle asserts that there exist points $\mathbf{x}_M$ and $\mathbf{x}_{\boldsymbol{m}}$ on bdy D such that

$$u\left(\mathbf{x}_m\right) \leq u(\mathbf{x}) \leq u\left(\mathbf{x}_{\boldsymbol{M}}\right)$$

for all $\mathbf{x} \in D$. Also, there are no points inside D with this property (unless $u \equiv$ constant). There could be several such points on the boundary.

The idea of the maximum principle is as follows, in two dimensions, say. At a maximum point inside D, if there were one, we'd have $u_{xx} \leq 0$ and $u_{yy} \leq 0$. (This is the second derivative test of calculus.) So $u_{xx}+u_{yy} \leq 0$. At most maximum points, $u_{xx}<0$ and $u_{yy}<0$. So we'd get a contradiction to Laplace's equation. However, since it is possible that $u_{xx}=0=u_{yy}$ at a maximum point, we have to work a little harder to get a proof.

Here we go. Let $\epsilon>0$. Let $v(\mathbf{x})=u(\mathbf{x})+\epsilon|\mathbf{x}|^2$. Then, still in two dimensions, say,

$$\Delta v=\Delta u+\epsilon \Delta\left(x^2+y^2\right)=0+4 \epsilon>0 \quad \text { in } D .$$

But $\Delta v=v_{xx}+v_{yy} \leq 0$ at an interior maximum point, by the second derivative test in calculus! Therefore, $v(\mathbf{x})$ has no interior maximum in D.

Now $v(\mathbf{x})$, being a continuous function, has to have a maximum somewhere in the closure $\bar{D}=D \cup$ bdy D. Say that the maximum of $v(\mathbf{x})$ is attained at $\mathbf{x}_0 \in$ bdy D. Then, for all $\mathbf{x} \in D$,

$$u(\mathbf{x}) \leq v(\mathbf{x}) \leq v\left(\mathbf{x}_0\right)=u\left(\mathbf{x}_0\right)+\epsilon\left|\mathbf{x}_0\right|^2 \leq \max _{\text {bdy } D} u+\epsilon l^2,$$

where l is the greatest distance from bdy D to the origin. Since this is true for any $\epsilon>0$, we have

$$u(\mathbf{x}) \leq \max _{\operatorname{bdy} D} u \quad \text { for all } \mathbf{x} \in D$$

Now this maximum is attained at some point $\mathbf{x}_M \in$ bdy D. So $u(\mathbf{x}) \leq u\left(\mathbf{x}_M\right)$ for all $\mathbf{x} \in \bar{D}$, which is the desired conclusion.

The existence of a minimum point x_m is similarly demonstrated. (The absence of such points inside D will be proved by the following) $\square$

Proof. **(2-D Strong)**

Here is a complete proof of its strong form. Let $u(\mathbf{x})$ be harmonic in D. The maximum is attained somewhere (by the continuity of u on $\bar{D}$), say at $\mathbf{x}_M \in \bar{D}$. We have to show that $\mathbf{x}_M \notin D$ unless $u \equiv$ constant. By definition of M, we know that

$$u(\mathbf{x}) \leq u\left(\mathbf{x}_M\right) = M \quad \text{ for all } \mathbf{x} \in D.$$

We draw a circle around $\mathbf{x}_M$ entirely contained in D. By the mean value property, $u\left(\mathbf{x}_M\right)$ is equal to its average around the circumference. Since the average is no greater than the maximum, we have the string of inequalities

$$M = u\left(\mathbf{x}_M\right) = \text{ average on circle } \leq M.$$

Therefore, $u(\mathbf{x}) = M$ for all $\mathbf{x}$ on the circumference. This is true for any such circle. So $u(\mathbf{x}) = M$ for all $\mathbf{x}$ in the diagonally shaded region (see Figure 3). Now we repeat the argument with a different center. We can fill the whole domain up with circles. In this way, using the assumption that D is connected, we deduce that $u(\mathbf{x}) \equiv M$ throughout D. So $u \equiv$ constant. $\square$

Remark: The n-D Strong Maximum Principle is called Hopf Maximum Principle which Strauss's book does not give a proof. [Page 181]

2.9 Uniqueness and Stability (Diffusion and Laplace)

- 1. Dirichlet Problem for the Diffusion Equation

Uniqueness:

Proof. **(maximum principle)**

The maximum principle can be used to give a proof of uniqueness for the Dirichlet problem for the diffusion equation. That is, there is at most one solution of

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) && \text{for } 0 < x < l \text{ and } t > 0 \\ u(x,0) &= \phi(x) \\ u(0,t) &= g(t) \qquad u(l,t) = h(t) \end{aligned}$$

for four given functions f, ϕ, g, and h. Uniqueness means that any solution is determined completely by its initial and boundary conditions. Indeed, let $u_1(x,t)$ and $u_2(x,t)$ be two solutions. Let $w = u_1 - u_2$ be their difference. Then $w_t - kw_{xx} = 0, w(x,0) = 0, w(0,t) = 0, w(l,t) = 0$. Let $T > 0$. By the maximum principle, $w(x,t)$ has its maximum for the rectangle on its bottom or sides-exactly where it vanishes. So $w(x,t) \leq 0$. The same type of argument for the minimum shows that $w(x,t) \geq 0$. Therefore, $w(x,t) \equiv 0$, so that $u_1(x,t) \equiv u_2(x,t)$ for all $t \geq 0$.

Stability [Page 45]:

L^2 sense: $\int_0^l [u_1(x,t) - u_2(x,t)]^2 \, dx \leq \int_0^l [\phi_1(x) - \phi_2(x)]^2 \, dx$

L^∞ sense: $\max_{0\leq x\leq l} |u_1(x,t) - u_2(x,t)| \leq \max_{0\leq x\leq l} |\phi_1(x) - \phi_2(x)|$ □

- 2. Neumann Problem for the Diffusion Equation [Ex. 2.4.15]
- 3. Dirichlet Problem for the Laplace Equation

Uniqueness (2-D) [Page 155-156]:

Proof. **(maximum principle)**

To prove the uniqueness, suppose that

$$\begin{aligned} \Delta u = f \quad &\text{in } D \qquad \Delta v = f \quad \text{in } D \\ u = h \quad &\text{on bdy } D \qquad v = h \quad \text{on bdy } D. \end{aligned}$$

We want to show that $u \equiv v$ in D. So we simply subtract equations and let $w = u - v$. Then $\Delta w = 0$ in D and $w = 0$ on bdy D. By the maximum principle

$$0 = w(\mathbf{x}_m) \leq w(\mathbf{x}) \leq w(\mathbf{x}_M) = 0 \quad \text{for all } \mathbf{x} \in D.$$

Therefore, both the maximum and minimum of $\boldsymbol{w}(\mathbf{x})$ are zero. This means that $w \equiv 0$ and $u \equiv v$. □

Uniqueness (n-D) [Page 182]:

Proof. (**energy method**)

If we have two harmonic functions u_1 and u_2 with the same boundary data, then their difference $u = u_1 - u_2$ is harmonic and has zero boundary data. We go back to Green's First Identity (29) and substitute $v = u$. Since u is harmonic, we have $\Delta u = 0$ and

$$\iint_{\text{bdy } D} u \frac{\partial u}{\partial n} dS = \iiint_D |\nabla u|^2 d\mathbf{x}.$$

Since $u = 0$ on bdy D, the left side of the above equation vanishes. Therefore, $\iiint_D |\nabla u|^2 d\mathbf{x} = 0$. By the first vanishing theorem in Section A.1, it follows that $|\nabla u|^2 \equiv 0$ in D. Now a function with vanishing gradient must be a constant (provided that D is connected). So $u(\mathbf{x}) \equiv C$ throughout D. But u vanishes somewhere (on bdy D), so C must be 0 . Thus $u(\mathbf{x}) \equiv 0$ in D. This proves the uniqueness of the Dirichlet problem. □

- 4. Neumann Problem for the Laplace Equation [Ex. 7.1.2]

2.10 Comparison of Wave and Diffusion Equation

Property		Waves	Diffusions
(i)	Speed of propagation?	Finite ($\leq c$)	Infinite
(ii)	Singularities for $t > 0$?	Transported along characteristics (speed $= c$)	Lost immediately
(iii)	Well-posed for $t > 0$?	Yes	Yes (at least for bounded solutions)
(iv)	Well-posed for $t < 0$?	Yes	No
(v)	Maximum principle	No	Yes
(vi)	Behavior as $t \to +\infty$?	Energy is constant so does not decay	Decays to zero (if ϕ integrable)
(vii)	Information	Transported	Lost gradually

Remark 1: As for property (i) for the diffusion equation, notice from formula (10) that the value of $u(x, t)$ depends on the values of the initial datum $\phi(y)$ for all y, where $-\infty < y < \infty$. Conversely, the value of ϕ at a point x_0 has an immediate effect everywhere (for $t > 0$), even though most of its effect is only for a short time near x_0. Therefore, the speed of propagation is infinite. Exercise

2.5.2(b) shows that solutions of the diffusion equation can travel at any speed. This is in stark contrast to the wave equation (and all hyperbolic equations).

Remark 2: As for (iv), there are several ways to see that the diffusion equation is not well-posed for $t < 0$ ("backward in time"). One way is the following. Let

$$u_n(x,t) = \frac{1}{n} \sin nx e^{-n^2 kt}.$$

You can check that this satisfies the diffusion equation for all x, t. Also, $u_n(x,0) = n^{-1} \sin nx \to 0$ uniformly as $n \to \infty$. But consider any $t < 0$, say $t = -1$. Then $u_n(x,-1) = n^{-1} \sin nx e^{+kn^2} \to \pm\infty$ uniformly as $n \to \infty$ except for a few x. Thus u_n is close to the zero solution at time $t = 0$ but not at time $t = -1$. This violates the stability, in the uniform sense at least.

Another way is to let $u(x,t) = S(x,t+1)$. This is a solution of the diffusion equation $u_t = ku_{xx}$ for $t > -1, -\infty < x < \infty$. But $u(0,t) \to \infty$ as $t \searrow -1$, as we saw above. So we cannot solve backwards in time with the perfectly nice-looking initial data $(4\pi k)^{-1} e^{-x^2/4}$.

Besides, any physicist knows that heat flow, Brownian motion, and so on, are irreversible processes. Going backward leads to chaos.

2.11 Fourier Series

2.11.1 As Representation of Solutions (Wave and Diffusion)

Remark: For homogeneous B.C.'s, we can use the method of **Separation of Variables:** $u(x,t) = X(x)T(t)$; For inhomogeneous B.C.'s, we can use the method of Expansion or Data Shifting [Page 147-150].

[**Homogeneous**]

- 1. Wave Equation with I.C. and Dirichlet B.C. on [0, ℓ] (Fourier Sine Series) (6) [Page 84-87]

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \tag{38}$$

where $\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$, $\psi(x) = \frac{n\pi c}{l} \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$;

Alternatively, $A_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx$, $B_n = \frac{2}{n\pi c} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx$.

Remark:

1. Frequency (coefficient of t): $\frac{n\pi c}{l} = \frac{n\pi\sqrt{T}}{l\sqrt{\rho}}$ [Page 87]
2. "Fundamental" note (take n=1): $\frac{\pi\sqrt{T}}{l\sqrt{\rho}}$ [Page 87]

- 2. Wave Equation with I.C. and Neumann B.C. on [0, ℓ] (Fourier Cosine Series) [Page 91]

$$\begin{cases} u_{tt} = c^2 u_{xx}, \text{ for } 0 < x < \ell \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), \text{ for } 0 < x < \ell \\ u_x(0,t) = u_x(\ell,t) = 0 \end{cases} \tag{39}$$

Solution:

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\cos\frac{n\pi x}{l} \tag{40}$$

where $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$, $\psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos\frac{n\pi x}{l}$;
Alternatively, $A_n = \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}dx$ (n ≥ 0), $B_n = \frac{2}{n\pi c}\int_0^l \psi(x)\cos\frac{n\pi x}{l}dx$ (n ≥ 1), $B_0 = \frac{2}{l}\int_0^l \psi(x)dx$

- 3. Diffusion Equation with I.C. and Dirichlet B.C. on [0, ℓ] (Fourier Sine Series) [Page 87-88]

$$\begin{cases} u_t = ku_{xx} \quad (0 < x < l, 0 < t < \infty) \\ u(x,0) = \phi(x) \\ u(0,t) = u(l,t) = 0 \end{cases} \tag{41}$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin\frac{n\pi x}{l} \tag{42}$$

where $\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$; Alternatively, $A_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}dx$.

- 4. Diffusion Equation with I.C. and Neumann B.C. on [0, ℓ] (Fourier Cosine Series) [Page 90]

$$\begin{cases} u_t = ku_{xx} \quad (0 < x < l, 0 < t < \infty) \\ u(x,0) = \phi(x) \\ u_x(0,t) = u_x(l,t) = 0 \end{cases} \tag{43}$$

Solution:

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos\frac{n\pi x}{l}. \tag{44}$$

where $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$; Alternatively, $A_n = \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}dx$ (n $\geq$ 0).

Remark: If we meet mixed B.C.'s, we need to use separation of variables to solve.

- 5. Wave/Diffusion Equation with I.C. and Robin B.C. on [0, ℓ] [See Section 4.3, Page 92-100]

- 6. Schrödinger Equation with I.C. and Neumann B.C. on [0, ℓ]

Solution:

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-i(n\pi/l)^2 t} \cos\frac{n\pi x}{l} \tag{45}$$

[**Inhomogeneous**]

7. Wave Equation with I.C. and Dirichlet B.C. on $[0,\ell]$ [Page 149, Ex. 5.6.11]

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \quad (0 < x < l, 0 < t < \infty) \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x) \\ u(0,t) = h(t), u(l,t) = k(t) \end{cases} \tag{46}$$

Solution:

$$u(x,t) = \frac{l-x}{l}h(t) + \frac{x}{l}k(t) + \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{l}, \quad 0 \le x \le l \tag{47}$$

where

$$\begin{aligned} a_n(t) = \frac{2}{n\pi}\{&[(-1)^n k(0) - h(0)]\cos\frac{cn\pi t}{l} + \frac{l}{cn\pi}[(-1)^n k'(0) - h'(0)]\sin\frac{cn\pi t}{l}\} \\ &+\frac{2}{l}\int_0^l \left[\phi(q)\sin\frac{n\pi q}{l}\cos\frac{cn\pi t}{l} + \frac{l}{cn\pi}\psi(q)\sin\frac{n\pi q}{l}\sin\frac{cn\pi t}{l}\right] dq \\ &+\frac{l}{cn\pi}\int_0^t \sin\left[\frac{cn\pi}{l}(t-s)\right]\left\{\frac{2}{l}\int_0^l f(q,s)\sin\frac{n\pi q}{l}dq + \frac{2}{n\pi}[(-1)^n k''(s) - h''(s)]\right\} ds \end{aligned}$$

8. Diffusion Equation with I.C. and Dirichlet B.C. on $[0,\ell]$ [Page 147-148]

$$\begin{cases} u_t = k u_{xx} \quad (0 < x < l, 0 < t < \infty) \\ u(x,0) = \phi(x) \\ u(0,t) = h(t), u(l,t) = j(t) \end{cases} \tag{48}$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{l} \tag{49}$$

where $u_n(t) = Ce^{-\lambda_n k t} - 2n\pi l^{-2}k\int_0^t e^{-\lambda_n k(t-s)}[(-1)^n j(s) - h(s)]\,ds$, with $\lambda_n = (n\pi/l)^2$

2.11.2 As Representation of Solutions (Laplace)

- 1. Laplace Equation with Specific (Dirichlet, Neumann, or Robin) B.C.'s on Rectangles/Cubes [See Section 6.2, Page 161-164]

Remark: Consider the symmetricity of the given B.C.'s as well as Maximum Principle. For the former, if the given B.C.'s on x-axis is symmetric, then we can apply Sufficient Condition Theorem (2.11.6) to exclude negative eigenvalue for the operator $-\frac{d}{dx^2}$. For the latter, if all the B.C.'s are Dirichlet B.C., then the solution should be a constant.

Example 1. Solve the Laplace Equation with the B.C.'s indicated in Figure (1). If we call the

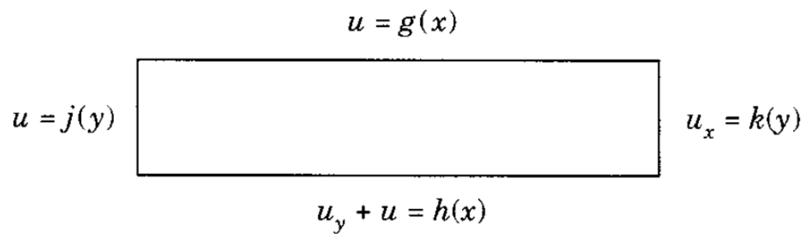

Figure 1

solution u with data (g, h, j, k), then $u = u_1 + u_2 + u_3 + u_4$ where u_1 has data $(g, 0, 0, 0)$, u_2 has data $(0, h, 0, 0)$, and so on. For simplicity, let's assume that $h = 0, j = 0$, and $k = 0$. Now we separate variables $u(x, y) = X(x) \cdot Y(y)$. We get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Hence there is a constant λ such that $X'' + \lambda X = 0$ for $0 \le x \le a$ and $Y'' - \lambda Y = 0$ for $0 \le y \le b$. Thus $X(x)$ satisfies a homogeneous one-dimensional problem which we well know how to solve: $X(0) = X'(a) = 0$. The solutions are

$$\beta_n^2 = \lambda_n = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{a^2} \quad (n = 0, 1, 2, 3, \ldots) \tag{50}$$

$$X_n(x) = \sin \frac{\left(n + \frac{1}{2}\right) \pi x}{a} \tag{51}$$

Next we look at the y variable. We have

$$Y'' - \lambda Y = 0 \quad \text{with } Y'(0) + Y(0) = 0.$$

(We shall save the inhomogeneous B.C.'s for the last step.) From the previous part, we know that $\lambda = \lambda_n > 0$ for some n. The Y equation has exponential solutions. As usual it is convenient to write them as

$$Y(y) = A\cosh\beta_n y + B\sinh\beta_n y.$$

So $0 = Y'(0) + Y(0) = B\beta_n + A$. Without losing any information we may pick $B = -1$, so that $A = \beta_n$. Then

$$Y(y) = \beta_n\cosh\beta_n y - \sinh\beta_n y.$$

Because we're in the rectangle, this function is bounded. Therefore, the sum

$$u(x,y) = \sum_{n=0}^{\infty} A_n \sin\beta_n x\left(\beta_n\cosh\beta_n y - \sinh\beta_n y\right) \tag{52}$$

is a harmonic function in D that satisfies all three homogeneous B.C.'s. The remaining B.C. is $u(x,b) = g(x)$. It requires that

$$g(x) = \sum_{n=0}^{\infty} A_n\left(\beta_n\cosh\beta_n b - \sinh\beta_n b\right)\cdot\sin\beta_n x$$

for $0 < x < a$. This is simply a Fourier series in the eigenfunctions $\sin\beta_n x$. By Chapter 5, the coefficients are given by the formula

$$A_n = \frac{2}{a}\left(\beta_n\cosh\beta_n b - \sinh\beta_n b\right)^{-1}\int_0^a g(x)\sin\beta_n x dx \tag{53}$$

- 2. Laplace Equation with Inhomogeneous Dirichlet B.C. on a Disk [Page 165-168]

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{for } x^2 + y^2 < a^2 \\ u = h(\theta) \quad \text{for } x^2 + y^2 = a^2 \end{cases} \tag{54}$$

Solution 1:

$$\begin{aligned} u(r,\theta) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n\left(A_n\cos n\theta + B_n\sin n\theta\right) \\ &= \left(a^2 - r^2\right)\int_0^{2\pi}\frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\frac{d\phi}{2\pi} \quad \textbf{[Poisson's Formula 1]} \end{aligned} \tag{55}$$

where $A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos n\phi\, d\phi$, $B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin n\phi\, d\phi$

Solution 2:

$$u(\mathbf{x})=\frac{a^2-|\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{x}'|=a}\frac{u(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}ds' \quad \textbf{[Poisson's Formula 2]} \tag{56}$$

Theorem 2.11.1. Let $h(\phi)=u(\mathbf{x}')$ be any continuous function on the circle $C=\text{bdy } D$. Then the Poisson formula (55), or (56), provides the only harmonic function in D for which

$$\lim_{\mathbf{x}\to\mathbf{x}_0} u(\mathbf{x})=h(\mathbf{x}_0) \qquad \text{for all } \mathbf{x}_0\in C. \tag{57}$$

This means that $u(\mathbf{x})$ is a continuous function on $\bar{D}=D\cup C$. It is also differentiable to all orders inside D.

Proof. We begin the proof by writing (55) in the form

$$u(r,\theta)=\int_0^{2\pi}P(r,\theta-\phi)h(\phi)\frac{d\phi}{2\pi}$$

for $r<a$, where

$$P(r,\theta)=\frac{a^2-r^2}{a^2-2ar\cos\theta+r^2}=1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n\theta \tag{58}$$

is the **Poisson kernel**. Note that P has the following three properties.

(i) $P(r,\theta)>0$ for $r<a$. This property follows from the observation that $a^2-2ar\cos\theta+r^2\geq a^2-2ar+r^2=(a-r)^2>0$.

(ii)

$$\int_0^{2\pi}P(r,\theta)\frac{d\theta}{2\pi}=1$$

This property follows from the second part of (58) because $\int_0^{2\pi}\cos n\theta d\theta=0$ for $n=1,2,\ldots$

(iii) $P(r,\theta)$ is a harmonic function inside the circle. This property follows from the fact that each term $(r/a)^n\cos n\theta$ in the series is harmonic and therefore so is the sum.

Now we can differentiate under the integral sign (as in Appendix A.3) to get

$$\begin{aligned} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta} &= \int_0^{2\pi}\left(P_{rr}+\frac{1}{r}P_r+\frac{1}{r^2}P_{\theta\theta}\right)(r,\theta-\phi)h(\phi)\frac{d\phi}{2\pi} \\ &= \int_0^{2\pi}0\cdot h(\phi)d\phi=0 \end{aligned}$$

for $r<a$. So u is harmonic in D. So it remains to prove (57). To do that, fix an angle θ_0 and

consider a radius r near a. Then we will estimate the difference

$$u(r, \theta_0) - h(\theta_0) = \int_0^{2\pi} P(r, \theta_0 - \phi)[h(\phi) - h(\theta_0)] \frac{d\phi}{2\pi} \tag{59}$$

by Property (ii) of P. But $P(r, \theta)$ is concentrated near $\theta = 0$. This is true in the precise sense that, for $\delta \leq \theta \leq 2\pi - \delta$,

$$|P(r, \theta)| = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = \frac{a^2 - r^2}{(a - r)^2 + 4ar\sin^2(\theta/2)} < \epsilon \tag{60}$$

for r sufficiently close to a. Precisely, for each (small) $\delta > 0$ and each (small) $\epsilon > 0$, (61) is true for r sufficiently close to a. Now from Property (i), (59), and (60), we have

$$|u(r, \theta_0) - h(\theta_0)| \leq \int_{\theta_0 - \delta}^{\theta_0 + \delta} P(r, \theta_0 - \phi)\, \epsilon \frac{d\phi}{2\pi} + \epsilon \int_{|\phi - \theta_0| > \delta} |h(\phi) - h(\theta_0)| \frac{d\phi}{2\pi} \tag{61}$$

for r sufficiently close to a. The ϵ in the first integral came from the continuity of h. In fact, there is some $\delta > 0$ such that $|h(\phi) - h(\theta_0)| < \epsilon$ for $|\phi - \theta_0| < \delta$. Since the function $|h| \leq H$ for some constant H, and in view of Property (ii), we deduce from (61) that

$$|u(r, \theta_0) - h(\theta_0)| \leq (1 + 2H)\epsilon$$

provided r is sufficiently close to a. This is relation (57). □

- 3. Laplace Equation with Homogeneous Dirichlet B.C. and Inhomogeneous Neumann B.C. on the Wedge.

[See Page 172-173]

- 4. Laplace Equation with Inhomogeneous Dirichlet B.C. on an Annulus

[See Page 174-175]

2.11.3 Fourier Sine, Cosine, Full Series

- Fourier Sine Series ($0 < x < \ell$)

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$
$$A_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx$$

- Fourier Cosine Series ($0 < x < \ell$)

$$\phi(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos \tfrac{n\pi x}{l}.$$
$$A_n = \tfrac{2}{l}\int_0^l \phi(x) \cos \tfrac{n\pi x}{l} dx.$$

- Fourier Full Series ($-\ell < x < \ell$)

Real Version:

$$\begin{aligned} \phi(x) &= \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \tfrac{n\pi x}{l} + B_n \sin \tfrac{n\pi x}{l}\right). \\ A_n &= \tfrac{1}{l}\int_{-l}^{l} \phi(x) \cos \tfrac{n\pi x}{l} dx \quad (n = 0, 1, 2, \ldots) \\ B_n &= \tfrac{1}{l}\int_{-l}^{l} \phi(x) \sin \tfrac{n\pi x}{l} dx \quad (n = 1, 2, 3, \ldots) \end{aligned}$$

Complex Version:

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}.$$
$$c_n = \tfrac{1}{2l}\int_{-l}^{l} \phi(x) e^{-in\pi x/l} dx.$$

- Examples of specific $\phi(x)$ [Page 108-111]

Remark:

1. Fourier Sine Series and Cosine Series on ($0 < x < \ell$) are the building blocks of Fourier Full Series on ($-\ell < x < \ell$). If the function is determined to be odd, then its Fourier Full Series is Sine Series; If the function is determined to be even, then its Fourier Full Series is Cosine Series; If the function is not determined to be odd or even, for instance, it is only defined on $\mathbb{R}^+$, then it can either be Sine or Cosine Series but with different B.C.'s.

2. The main problem for Fourier Series representation of the solutions is the B.C. It only works for symmetric B.C.'s, otherwise the sine and cosine elements are not orthogonal anymore.

3. Fourier Full Series converges to any L^2 function pointwisely almost everywhere. For those failed points with measure zero, for instance as continuous function with countably many jump discontinuities, it is the average value at each jump (2.11.15).

2.11.4 Criteria in the Method of Separation of Variables

1-D Green's Second Identity: $(\lambda_1 - \lambda_2)\int_a^b X_1X_2\,dx = \int_a^b (X_1X_2'' - X_1''X_2)\,dx = (X_1X_2' - X_1'X_2)|_a^b$ where $X_1(x)$ and $X_2(x)$ are two different eigenfunctions in the approach of separation of variables:

$$-X_1'' = \frac{-d^2X_1}{dx^2} = \lambda_1 X_1$$
$$-X_2'' = \frac{-d^2X_2}{dx^2} = \lambda_2 X_2$$

Definition 2.11.2. (1-D) A boundary condition is called **symmetric for the operator -$\frac{d}{dx^2}$** if $X_1'(x)X_2(x) - X_1(x)X_2'(x)|_{x=a}^{x=b} = 0$ [Page 119]

Recall: n-D Green's Second Identity: $\iiint_D (u\Delta v - v\Delta u)d\mathbf{x} = \iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$

Definition 2.11.3. (n-D) A boundary condition is called **symmetric for the operator Δ** if $\iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS = 0$ [Page 185]

Remark:

1. n-D covers 1-D since $\frac{\partial v}{\partial n}$ includes a unit outer normal vector
2. Periodic ($X_i(a) = X_i(b), X_i'(a) = X_i'(b)$), homogeneous Dirichlet, Neumann, Robin B.C.'s are symmetric. Inhomogeneous Dirichlet, Neumann, Robin B.C.'s are not symmetric. Then separation of variables can not work.

Theorem 2.11.4. If we have symmetric boundary conditions, then any two eigenfunctions that correspond to distinct eigenvalues are orthogonal. Therefore, if any function is expanded in a series of these eigenfunctions, the coefficients are determined. [Page 120]

Proof. Take two different eigenfunctions $X_1(x)$ and $X_2(x)$ with $\lambda_1 \neq \lambda_2$. We write Green's second identity (3). Because the boundary conditions are symmetric, the right side of (3) vanishes. Because of the different equations, the identity takes the form (3a), and the orthogonality is proven. If $X_n(x)$ now denotes the eigenfunction with eigenvalue λ_n and if

$$\phi(x) = \sum_n A_n X_n(x)$$

is a convergent series, where the A_n are constants, then

$$(\phi, X_m) = \left(\sum_n A_n X_n, X_m\right) = \sum_n A_n (X_n, X_m) = A_m (X_m, X_m)$$

by the orthogonality. So if we denote $c_m = (X_m, X_m)$, we have

$$A_m = \frac{(\phi, X_m)}{c_m}$$

as the formula for the coefficients. □

Remark: If there are two eigenfunctions, say $X_1(x)$ and $X_2(x)$, but their eigenvalues are the same, $\lambda_1 = \lambda_2$, then they don't have to be orthogonal. But if they aren't orthogonal, they can be made so by the Gram-Schmidt orthogonalization procedure (Ex. 5.3.10).

Theorem 2.11.5. If we have symmetric boundary conditions, then all the eigenvalues are real numbers. Furthermore, all the eigenfunctions can be chosen to be real valued. [Page 121]

Proof. Let λ be an eigenvalue, possibly complex. Let $X(x)$ be its eigenfunction, also possibly complex. Then $-X'' = \lambda X$ plus B.C.'s. Take the complex conjugate of this equation; thus $-\bar{X}'' = \bar{\lambda}\bar{X}$ plus B.C.'s. So $\bar{\lambda}$ is also an eigenvalue. Now use Green's second identity with the functions X and $\bar{X}$. Thus

$$\int_a^b \left(-X''\bar{X} + X\bar{X}''\right) dx = \left(-X'\bar{X} + X\bar{X}'\right)\Big|_a^b = 0$$

since the B.C.'s are symmetric. So

$$(\lambda - \bar{\lambda}) \int_a^b X\bar{X} dx = 0$$

But $X\bar{X} = |X|^2 \geq 0$ and $X(x)$ is not allowed to be the zero function. So the integral cannot vanish. Therefore, $\lambda - \bar{\lambda} = 0$, which means exactly that λ is real.

Next, let's reconsider the same problem $-X'' = \lambda X$ together with (4), knowing that λ is real. If $X(x)$ is complex, we write it as $X(x) = Y(x) + iZ(x)$, where $Y(x)$ and $Z(x)$ are real. Then $-Y'' - iZ'' = \lambda Y + i\lambda Z$. Equating the real and imaginary parts, we see that $-Y'' = \lambda Y$ and $-Z'' = \lambda Z$. The boundary conditions still hold for both Y and Z because the eight constants in (4) are real numbers. So the real eigenvalue λ has the real eigenfunctions Y and Z. We could therefore say that X and $\bar{X}$ are replaceable by the Y and Z. The linear combinations $aX + b\bar{X}$ are the same as the linear combinations $cY + dZ$, where a and b are somehow related to c and d. This completes the proof. □

Theorem 2.11.6. Suppose we have symmetric boundary conditions for the operator -$\frac{d}{dx^2}$. If

$$f(x)f'(x)|_{x=a}^{x=b} \leq 0$$

for all (real-valued) functions $f(x)$ satisfying the B.C.'s, then there is no negative eigenvalue. [Page 122]

Proof. Take f(x)=X(x).

As Green's Second Identity: $\int_a^b X''(x)X(x)dx = X(x)X'(x)|_a^b - \int_a^b [X'(x)]^2\,dx$
Take $X'' = -\lambda X(x)$ and by the condition we have, we get $\int_a^b(-\lambda)[X(x)]^2dx + \int_a^b [X'(x)]^2\,dx \le 0$.
Thus $\lambda \ge 0$. □

Remark: This is only a sufficient condition for determining the sign of the eigenvalues. It is easy to verify that this sufficient condition is valid for periodical, homogeneous Dirichlet, Neumann B.C.'s, so that in these cases there are no negative eigenvalues]. However, this sufficient condition could not be valid for certain Robin boundary conditions. [See Ex.5.3.11]

2.11.5 Completeness of the Fourier Series

Theorem 2.11.7. There are an infinite number of eigenvalues. They form a sequence $\lambda_n \to +\infty$. [Page 125]

Definition 2.11.8. We say that an infinite series $\Sigma_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ **pointwise** in (a,b) if it converges to $f(x)$ for each $a < x < b$. That is, for each $a < x < b$ we have

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as } N \to \infty.$$

Definition 2.11.9. We say that the series converges **uniformly** to $f(x)$ in $[a,b]$ if

$$\max_{a \le x \le b} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as } N \to \infty.$$

Remark: Note that the endpoints are included in this definition. That is, you take the biggest difference over all the x 's and then take the limit.

Definition 2.11.10. We say the series converges in the **mean-square** (or L^2) sense to $f(x)$ in (a,b) if

$$\int_a^b \left| f(x) - \sum_{n=1}^{N} f_n(x) \right|^2 dx \to 0 \quad \text{as } N \to \infty. \tag{62}$$

- See examples at [Page 126-128].

Theorem 2.11.11. Uniform Convergence of Fourier Series [Page 128]
The Fourier series $\Sigma A_n X_n(x)$ converges to $f(x)$ uniformly on $[a,b]$ provided that:
(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and

(ii) $f(x)$ satisfies the given boundary conditions.

Remark: This theorem assures us of a very good kind of convergence provided that the conditions on $f(x)$ and its derivatives are met. For the classical Fourier series (full, sine, and cosine), it is not required that $f''(x)$ exist.

Proof. We assume again that $f(x)$ and $f'(x)$ are continuous functions of period 2π. The idea of this proof is quite different from the preceding one. The main point is to show that the coefficients go to zero pretty fast. Let A_n and B_n be the Fourier coefficients of $f(x)$ and let A_n' and B_n' denote the Fourier coefficients of $f'(x)$. We integrate by parts to get

$$\begin{aligned} A_n &= \int_{-\pi}^{\pi} f(x) \cos nx \frac{dx}{\pi} \\ &= \frac{1}{n\pi} f(x) \sin nx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x) \sin nx \frac{dx}{n\pi}, \end{aligned}$$

so that

$$A_n = -\frac{1}{n} B_n' \quad \text{for n } \neq 0.$$

We have just used the periodicity of $f(x)$. Similarly,

$$B_n = \frac{1}{n} A_n'.$$

On the other hand, we know from Bessel's inequality [for the derivative $f'(x)$] that the infinite series

$$\sum_{n=1}^{\infty} \left(|A_n'|^2 + |B_n'|^2 \right) < \infty$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} (|A_n \cos nx| + |B_n \sin nx|) &\leq \sum_{n=1}^{\infty} (|A_n| + |B_n|) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} (|B_n'| + |A_n'|) \\ &\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left[\sum_{n=1}^{\infty} 2 \left(|A_n'|^2 + |B_n'|^2 \right) \right]^{1/2} < \infty \end{aligned}$$

Here we have used Schwartz's inequality (see Exercise 5). The result means that the Fourier series converges absolutely.

We already know (from Theorem (2.11.16)) that the sum of the Fourier series is indeed $f(x)$. So, again denoting by $S_N(x)$ the partial sum (2), we can write

$$\max|f(x) - S_N(x)| \leq \max \sum_{n=N+1}^{\infty} |A_n \cos nx + B_n \sin nx|$$
$$\leq \sum_{n=N+1}^{\infty} (|A_n| + |B_n|) < \infty.$$

The last sum is the tail of a convergent series of numbers so that it tends to zero as $N \to \infty$. Therefore, the Fourier series converges to $f(x)$ both absolutely and uniformly.

This proof is also valid if $f(x)$ is continuous but $f'(x)$ is merely piecewise continuous. An example is $f(x) = |x|$. □

Theorem 2.11.12. L^2 **Convergence of Fourier Series** [Page 128]

The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2 dx \text{ is finite.}$$

Remark: This theorem assures us a weaker convergence. We can extend the integral to Lebesgue Integral.

Definition 2.11.13. A function $f(x)$ has a **jump discontinuity** at a point $x = c$ if the one-sided limits $f(c+)$ and $f(c-)$ exist but are not equal. [It doesn't matter what $f(c)$ happens to be or even whether $f(c)$ is defined or not.] The value of the jump discontinuity is the number $f(c+) - f(c-)$.

Definition 2.11.14. A function f (x) is called **piecewise continuous** on an interval [a, b] if it is continuous at all but a finite number of points and has jump discontinuities at those points.

Theorem 2.11.15. Pointwise Convergence of Fourier Series [Page 129]

(i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b) provided that $f(x)$ is a continuous function on $a \leq x \leq b$ and $f'(x)$ is piecewise continuous on $a \leq x \leq b$.

(ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \leq x \leq b$ and $f'(x)$ is also piecewise continuous on $a \leq x \leq b$, then the classical Fourier series converges at every point $x(-\infty < x < \infty)$. The sum is

$$\sum_n A_n X_n(x) = \frac{1}{2}[f(x+) + f(x-)] \quad \text{for all } a < x < b.$$

Remark: The sum is $\frac{1}{2}\left[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)\right]$ for all $-\infty < x < \infty$, where $f_{\text{ext}}(x)$ is the extended function (periodic, odd periodic, or even periodic).

Theorem 2.11.16. Extension of 2.11.15 [Page 129]
If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for $-\infty < x < \infty$.

Remark: This is the extension of the previous theorem.

Proof. See Section 5.5 with the proof that starts from 2π periodical C^1 functions to piecewise continuous by modification. [Page 136-140] □

Theorem 2.11.17. Least-Square Approximation [Page 131-132]
Let $\{X_n\}$ be any orthogonal set of functions. Let $\|f\| < \infty$. Let N be a fixed positive integer. Among all possible choices of N constants $c_1, c_2, \ldots, c_N$, the choice that minimizes

$$\left\| f - \sum_{n=1}^{N} c_n X_n \right\|$$

is $c_1 = A_1, \ldots, c_n = A_n$, where A_n are Fourier coefficients.

Proof. For the sake of simplicity we assume in this proof that $f(x)$ and all the $X_n(x)$ are real valued. Denote the error (remainder) by

$$E_N = \left\| f - \sum_{n \le N} c_n X_n \right\|^2 = \int_a^b \left| f(x) - \sum_{n \le N} c_n X_n(x) \right|^2 dx.$$

Expanding the square, we have (assuming the functions are real valued)

$$\begin{aligned} E_N = &\int_a^b |f(x)|^2 dx - 2 \sum_{n \le N} c_n \int_a^b f(x) X_n(x) dx \\ &+ \sum_n \sum_m c_n c_m \int_a^b X_n(x) X_m(x) dx. \end{aligned}$$

Because of orthogonality, the last integral vanishes except for $n = m$. So the double sum reduces to $\Sigma c_n^2 \int |X_n|^2 dx$. Let us write this in the norm notation:

$$E_N = \|f\|^2 - 2 \sum_{n \le N} c_n (f, X_n) + \sum_{n \le N} c_n^2 \|X_n\|^2 \tag{63}$$

We may "complete the square":

$$E_N = \sum_{n\le N} \|X_n\|^2 \left[c_n - \frac{(f, X_n)}{\|X_n\|^2} \right]^2 + \|f\|^2 - \sum_{n\le N} \frac{(f, X_n)^2}{\|X_n\|^2}.$$

Now the coefficients c_n appear in only one place, inside the squared term. The expression is clearly smallest if the squared term vanishes. That is,

$$c_n = \frac{(f, X_n)}{\|X_n\|^2} \equiv A_n$$

which proves the theorem. □

Theorem 2.11.18. Bessel's Inequality [Page 132-133] For $f \in L^2$,

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 \, dx \le \int_a^b |f(x)|^2 dx \tag{64}$$

Proof. Take $c_n = A_n$ in (63) to prove the Bessel's Inequality. □

Theorem 2.11.19. Parseval's Equality [Page 133]

The Fourier series of $f(x)$ converges to $f(x)$ in the mean-square (L^2) sense if and only if

$$\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 \, dx = \int_a^b |f(x)|^2 dx \tag{65}$$

Proof. L^2 convergence means that the remainder $E_N \to 0$. But from (63) this means that $\Sigma_{n\le N} |A_n|^2 \|X_n\|^2 \to \|f\|^2$, which in turn means (65). □

Definition 2.11.20. The infinite orthogonal set of functions $\{X_1(x), X_2(x), \ldots\}$ is called **complete** if Parseval's equality (65) is true for all f with $\int_a^b |f|^2 dx < \infty$.

Definition 2.11.21. Dirichlet Kernel

$$K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos n\theta \overset{\text{(i)}}{=} \frac{\sin\left(N + \frac{1}{2}\right)\theta}{\sin \frac{1}{2}\theta} \tag{66}$$

Remark:

1. The Dirichlet Kernel is derived from direct substitution of Fourier coefficients into the Fourier Series of 2π periodical C^1 functions. [See Page 137]

2. $\int_{-\pi}^{\pi} \frac{1}{2\pi} K_N(\theta) \, d\theta = 1$

3. $\frac{1}{2\pi} K_N(\theta) \xrightarrow{N\to\infty} \delta(\theta)$

Proof. Proof of **(i)**

The easiest proof is by complexification. By De Moivre's formula for complex exponentials,

$$K_N(\theta) = 1 + \sum_{n=1}^{N} \left(e^{in\theta} + e^{-in\theta}\right) = \sum_{n=-N}^{N} e^{in\theta}$$
$$= e^{-iN\theta} + \cdots + 1 + \cdots + e^{iN\theta}.$$

This is a finite geometric series with the first term $e^{-iN\theta}$, the ratio $e^{i\theta}$, and the last term $e^{iN\theta}$. So it adds up to

$$\begin{aligned} K_N(\theta) &= \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} \\ &= \frac{e^{-i\left(N+\frac{1}{2}\right)\theta} - e^{+i\left(N+\frac{1}{2}\right)\theta}}{-e^{\frac{1}{2}i\theta} + e^{-\frac{1}{2}i\theta}} \\ &= \frac{\sin\left[\left(N+\frac{1}{2}\right)\theta\right]}{\sin\frac{1}{2}\theta} \end{aligned}$$

□

- See The Gibbs Phenomenon at [Page 142-144]

2.11.6 Validation of Fourier Series Representation of PDE Solutions

Consider the Homogeneous Wave Equation with I.C. and Dirichlet B.C. (6). The solution is supposed to be:

$$u(x,t) = \sum_n \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}. \tag{67}$$

However, we know that **term-by-term differentiation of a Fourier series is not always valid** [See Example 3, Section 5.4, Page 130], so we cannot simply verify by direct differentiation that (67) is a solution.

Instead, let ϕ_{ext} and ψ_{ext} denote the odd $2l$-periodic extensions of ϕ and ψ. Let us assume that ϕ and ψ are continuous with piecewise continuous derivatives. We know that the function

$$u(x,t) = \frac{1}{2}\left[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)ds$$

solves the wave equation with $u(x,0) = \phi_{\text{ext}}(x), u_t(x,0) = \psi_{\text{ext}}(x)$ for all $-\infty < x < \infty$. Since ϕ_{ext} and ψ_{ext} agree with ϕ and ψ on the interval $(0,l)$, u satisfies the correct initial conditions on $(0,l)$. Since ϕ_{ext} and ψ_{ext} are odd, it follows that $u(x,t)$ is also odd, so that $u(0,t) = u(l,t) = 0$, which is the correct boundary condition.

By Theorem 2.11.15, the Fourier sine series of ϕ_{ext} and ψ_{ext}, given by (7), converge pointwise. Substituting these series into (67), we get

$$\begin{aligned} u(x,t) =& \frac{1}{2}\sum_{n=1}^{\infty} A_n \left(\sin\frac{n\pi(x+ct)}{l} + \sin\frac{n\pi(x-ct)}{l}\right) \\ &+ \frac{1}{2c}\sum_{n=1}^{\infty}\int_{x-ct}^{x+ct} B_n \frac{n\pi c}{l}\sin\frac{n\pi s}{l} ds. \end{aligned}$$

This series converges pointwise because **term-by-term integration of a Fourier series is always valid** [Ex. 5.4.11]. Now we use standard trigonometric identities and carry out the integrals explicitly. We get

$$u(x,t) = \sum_n \left(A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}\right).$$

This is precisely (67).

References

[1] Walter A. Strauss. Partial differential equations: an introduction. Wiley, New York, 2nd ed edition, 2009.