Unveiling the Roots of Matrices: A Generalization and Field Extension of the Square Root of Matrices

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I. Abstract

In this poster, we explore and investigate advanced topics in linear algebra. Specifically, we first prove the Theorem: *Any positive semi-definite symmetric real matrix has a unique positive semi-definite symmetric square root* by polynomial matrices. Then we give a generalization from field $\mathbb{R}$ to $\mathbb{C}$, and from square root to k-th root. Finally, we demonstrate the Theorem: *Any invertible complex matrix has a k-th root* by Jordan Canonical Form and Taylor Expansion. Our proofs' validity and theorems' value are considered and discussed. These topics have significant applications in the field of Algebra and Lie Theory.

II. Unique Existence of Real Square Root

Theorem 1 *Any positive semi-definite symmetric real matrix has a unique positive semi-definite symmetric square root.*

- Assumption:

$\forall A \in M_{n\times n}(\mathbb{R})$ & $A = A^T$ & A is positive semi-definite

Decompose $A = UDU^T$, where U is orthogonal and $D = diag(\lambda_1, \cdots, \lambda_n)$, all eigenvalues ($\lambda_i \in \mathbb{R}$) are nonnegative

1. Existence

Let $B = UD^{\frac{1}{2}}U^T$, where $D^{\frac{1}{2}} = diag\left(\lambda_1^{\frac{1}{2}}, \cdots, \lambda_n^{\frac{1}{2}}\right)$,

then $B^2 = A$ & $B = B^T$ & B is positive semi-definite

2. Uniqueness

a) $\exists$ a polynomial p with real coefficients $s.t.$

$p(A) = p(UDU^T)$

$= \sum_{i=0}^{n-1} a_i (UDU^T)^i = Up(D)U^T \triangleq UD^{\frac{1}{2}}U^T = B$

b) If $\exists$ C is another square root of A, then

$CA = AC$ and $CB = Cp(A)$

$$= \sum_{i=0}^{n-1} a_i CA^i = \sum_{i=0}^{n-1} a_i AC^i = p(A)C = BC$$

c) $\exists$ an orthogonal V that simultaneously diagonalizes B and C $i.e.$ $B = VD_1V^T$, $C = VD_2V^T$, where D_1, D_2 are real diagonal with nonnegative entries

d) ${D_1}^2 = {D_2}^2$ ($\because B^2 = A = C^2$), then $D_1 = D_2, B = C$ ■

III. Generalization

Theorem 2 *Any positive semi-definite Hermitian complex matrix has a unique positive semi-definite Hermitian k-th root.*

- Field: $\mathbb{R} \to \mathbb{C}$
- Root: $\sqrt{\ } \to \sqrt[k]{\ }$

IV. Field Extension for K-th Root Existence

Theorem 3 *Any invertible complex matrix has a k-th root.*

Proving Process:

- Assumption: $\forall T \in M_{n\times n}(\mathbb{C})$ 「1」 & *T is invertible* 「2」
- Goal: Precisely find the k-th root $R \in M_{n\times n}(\mathbb{C})$ $(k \geq 2)$

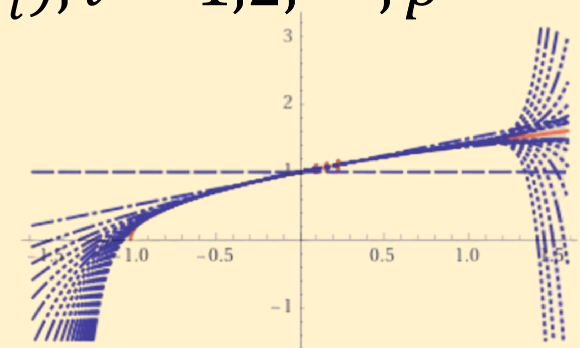
1. Jordan Canonical Form

$$\text{「1」}\ T \sim J \triangleq \begin{bmatrix} J(\lambda_1) & & \\ & \ddots & \\ & & J(\lambda_p) \end{bmatrix},\ J(\lambda_i) \triangleq \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

a) 「2」 Eigenvalues ($\lambda_i \in \mathbb{C}$) are nonzero

b) $N_i \triangleq \frac{J(0)}{\lambda_i}$, then $J(\lambda_i) = \lambda_i(I + N_i), i = 1,2,\cdots,p$

2. Taylor Expansion Theorem



$$\sqrt[k]{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{l=0}^{n}(lk-1)}{n!\,k^n(nk-1)} x^n, -1 < x < 1$$

a) Application in Matrix

$$\sqrt[k]{I + N_i} = \sum_{n=0}^{m} \frac{(-1)^n \prod_{l=0}^{n}(lk-1)}{n!\,k^n(nk-1)} {N_i}^n \triangleq Z_i$$

where $m = lub\{\, q \mid {N_i}^q = 0, q \in \mathbb{N}^* \,\} - 1$

b) Similarity Transitivity

$\therefore J(\lambda_i) = \lambda_i(I + N_i) = (\lambda_i^{\frac{1}{k}} Z_i)^k, i = 1,2,\cdots,p$

$$T = PJP^{-1} = (P \begin{bmatrix} \lambda_1^{\frac{1}{k}} Z_1 & & \\ & \ddots & \\ & & \lambda_p^{\frac{1}{k}} Z_p \end{bmatrix} P^{-1})^k \triangleq R^k$$ ■

3. Examples & Counterexamples

a) $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \exists R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s.t.\ R^k = T$

($\because$ T is invertible) 「2」

b) $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\nexists R\ \ s.t.\ R^k = T$ (T is singular)

c) $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \exists R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} s.t.\ R^k = T$

($t \in \mathbb{C}$) (T is singular)

V. Conclusion & Discussion

- Results
  - a) Positive semi-definite is nearly an extremely general condition for the uniqueness of the matrix roots.
  - b) Invertibility is just a sufficient condition for the existence of the matrix roots.
- Validity in Proofs
  - a) Theorem 2: The generalization of either the field (from $\mathbb{R} \to \mathbb{C}$) or the root (from $\sqrt{\ } \to \sqrt[k]{\ }$) does not cause any change in the proof of Theorem 1.
  - b) Taylor Expansion Theorem's Application in Matrix: N_i is a nilpotent matrix, which restricts "Matrix Taylor Series" into finite, thus convergent. Also, matrix retains the properties of real variable "x" in the Taylors Series.
- Value

  Lay a foundation for complicated matrix operations (functions)

VI. References

[1] Higham, N. J. Theories of Matrix Functions. In *Functions of matrices: Theory and computation*; SIAM: Philadelphia, PA, 2008; pp. 1–29.

[2] Horn, R. A.; Johnson, C. R. Positive definite and semidefinite matrices. In *Matrix analysis*; Cambridge University Press: New York, NY, 2017; pp. 439–440.